# Planetary motion around the Sun

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# 1 - The Copernican Revolution

In the first half of sixteenth century, the Polish astronomer Nicolaus Copernicus proposed a theory of planetary motion on a Sun centered system orbits, opposing it to the Ptolemaic conception, with the Earth in the center of solar system.

The idea was not new, already had been expressed by the Greeks, but Copernicus he gave a rigorous proof, using mathematical procedures. However his theory was not without defects.

Copernicus not fully exploited his idea of a planetary system centered on the Sun. He maintained, in his mathematical explanation, part of the Ptolemaic conception, so, in addition to still consider planetary circular orbits, he continued to think about orbital planes intersecting in the center of Earth's orbit.

Johannes Kepler perfected the Copernicus mathematical system, by applying strictly his own theory of a heliocentric universe. He, taking that the plans of planetary orbits were instead intersect in the sun, managed to transform the intricate Copernican system in a highly easy and accurate technique to calculate the position of the planets.

Kepler, who worked with Tycho Brahe, inherited from it a great amount of the most accurate data ever collected on the positions of the planets, and with these began to study to fix the Copernican theory. It was an huge job that occupy much of Kepler's time for nearly ten years. With a series of unsuccessful attempts, Kepler tried to calculate the orbit of Mars and that of the Earth, from which Mars was observed. To do this, he used a variety of combinations of circles and oval shapes, but none of these was able to eliminate inconsistencies between his theory and experimental observations. Then, casually, he found that theory and observations could get along if planets they move in elliptical orbits, with variable speed according to a simple law.

Kepler thought that to push the planets in their orbits were a driving force, the *anima motrix*, generated by the Sun, which was to decrease with increasing distance of planet from the Sun: to twice the distance should correspond an halved force, and therefore an orbital speed inversely proportional to the distance from the Sun. This law work well for small eccentricity, which were those of planetary orbits, and in the approximation of astronomical measurements at the time of Kepler, when the telescopes did not yet exist.

Isaac Newton formulated later the exact law: that of the force inversely proportional to the *square* of distance.

# 2 - An interesting hypothesis

The origins of previous quantitative deductions can be explained by the following *empirical* hypothesis:

Kepler, perhaps influenced by the continuous use of plane orbits in his

studies, must have considered the Sun action limited to the orbital plane, with the influence uniformly distributed on concentric circles centered on the Sun, so, doubling the radius the action would be halved.

Newton instead, free from any intellectual conditioning, considers realistically the Sun action uniformly distributed on concentric spherical surfaces centered on the Sun, so, on a sphere of double radius, with quadruple surface, the unitary influence would be reduced to a quarter.

I like to imagine that, having reached at some point of his creative path, Newton exclaimed:

*«The Sun attracts the planets as well as enlightens them!»*
formulating so in embryo that fantastic simple idea that would later lead it to the discovery of the Law of Universal Gravitation.

# 3 - The Newton's Synthesis

The Kepler setting introduced the physical concept of planetary motions regulated by forces, against the traditional view of "natural" circular motions. It was therefore necessary to "explain" the nature of Kepler's forces. Studies carried out later to this end, led to the Newtonian view of the universe, which gave to the Copernican revolution its final appearance.

In 1674 Robert Hooke gave a first qualitative description of phenomena governing celestial motions, postulating the existence of two basic agents: the *inertia*, already introduced by Descartes, which is the property of a body to resist to a change in its state of motion, and the *gravity force*, which is the mutual attraction force between any bodies.

Newton, who had already arrived alone at the qualitative conception of Hooke, he managed after to solve quantitatively the problem of planetary motion deducting two physical consequences of extreme importance. If the velocities of the planets and the rays of their orbits were bound together by Kepler's third law, then the pull of the Sun on the planets must decrease with inverse proportionality to the square of the distance of the planets from the sun. Furthermore, an inverse proportionality quadratic law could explain exactly both, the elliptic orbits of the first Kepler's law, either the velocity change described in the second law.

These mathematical deductions, that they had no precedent in the history of sciences, will be described in detail below.

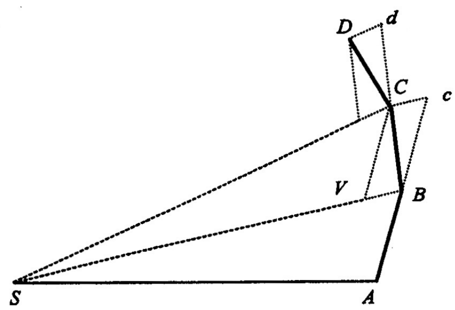
# 4 - The Feynman's lost lesson

Tycho Brahe, using a geocentric reference system, performed accurate measurements of the motion of Mars. His records, translated into a heliocentric reference, enabled Johannes Kepler to deduce his own laws of planetary motion, promoting the affirmation of the Copernican heliocentrism. Kepler said:

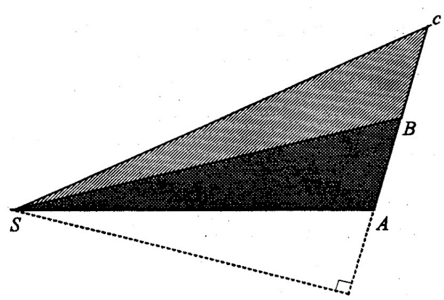
*1. The orbit of a planet is an ellipse with the Sun at one of the two foci.*
*2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.*
*3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.*

Later, Isaac Newton used Kepler's second law to show that the equality of the areas in equal times is equivalent to saying that the forces acting on the planets are directed towards the Sun. Its proof is contained in his book Principia.

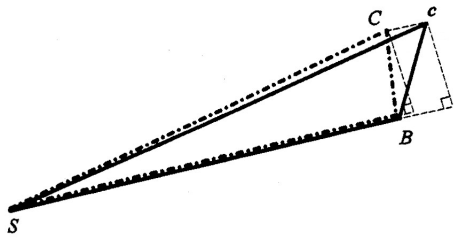
Newton plots the orbit with a set of inertial segments, interrupted by sharp changes in direction due to the application of the Sun's power for brief time intervals.




In the first time interval, the planet would move from $A$ to $B$, if there was no force acting on it. In the next equal length time interval, the planet would continue to move in a straight line from $B$ to $c$. A pulse of force, applied at point $B$, gives a motion component $BV$ directed toward the Sun. Composing, one has that the effective motion occurs along the diagonal $BC$. The same procedure repeats for each subsequent point. Newton shows that the planet orbit sweeps out equal areas in equal times. In other words, the $SAB$ triangle, swept by the planet in the first time interval, has the same area of the $SBC$ triangle, swept in the second time interval, and so on.

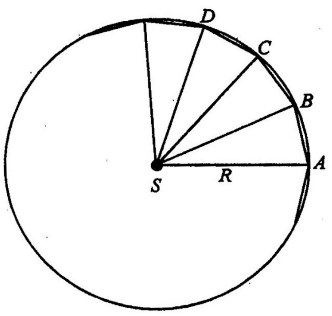


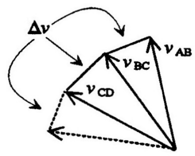
$SAB$ and $SBc$ triangles, forming the two areas, are equal, because they have equal basis and the same height.




$SBC$ and $SBc$ triangles have a common base and equal heights, because they are between parallel lines. Follows the equality of the $SAB$ and $SBC$ triangles, ie: ***planet sweeps out equal areas in equal times.***
Applying the same analysis to more and more short time intervals, you get a trajectory next as you want to a flat curve, on which both the inertia and the Sun attraction act continuously, producing an orbital motion in which the areas swept in equal times are equal. This result was obtained by assuming that the velocity variations, ie the forces, are directed towards the Sun.

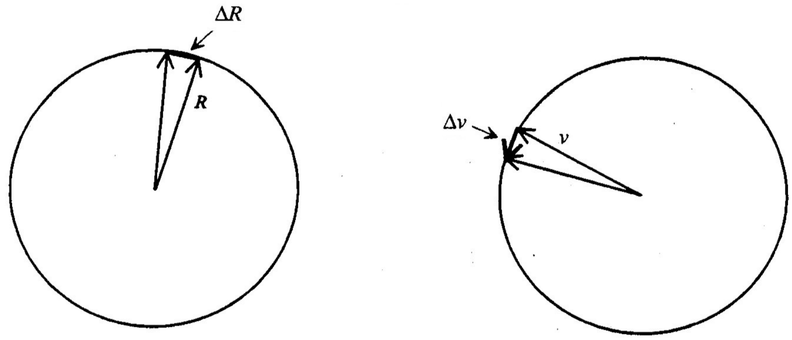
Newton then demonstrates, using Kepler's third law, that so directed velocity variations (forces) must vary as the inverse square of the distance. Let us assume that the orbit is simply a circle of radius $R$. Then the Newton diagram would look like this:




The orbit diagram is a regular polygon inscribed in a circumference which is the real orbit. To the right is drawn the relative velocity diagram, in which the $\Delta v$ variations form a regular polygon similar to the above, but rotated by 90°. If you take more and more short time intervals, both diagrams tend to the circumference, as in the following figures:

Since the force $F$ is proportional to $2\pi v/T$ and the velocity $v$ equal to $2\pi R/T$, we write:
$F \sim 2\pi v/T = (2\pi/T) \cdot (2\pi R/T) = 4\pi^2 R/T^2$
from Kepler's third law: $T \sim R^{3/2}$ and then: $T^2 \sim (R^{3/2})^2 = R^3$, so:



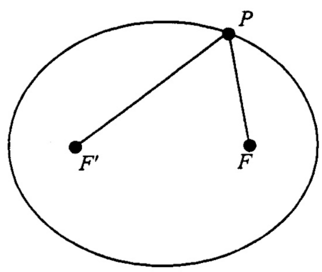

orbit diagram　　　　velocity diagram

$$\frac{\Delta R}{\Delta t} = \frac{2\pi R}{T} \qquad \frac{\Delta v}{\Delta t} = \frac{2\pi v}{T}$$

$F \sim R/T^2 \sim R/R^3 = 1/R^2$
ie, ***the force is proportional to the inverse square of the distance from the Sun.***

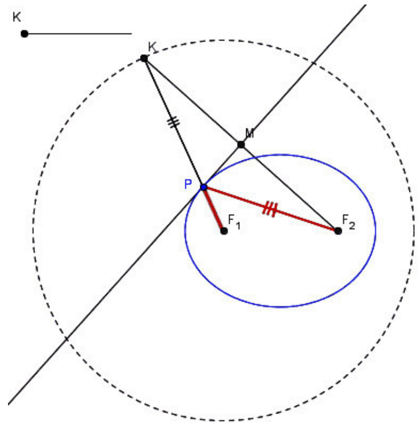
So we know that the force of gravity, exerted by the Sun on a planet, is directed towards the Sun and its intensity decreases as the inverse square of the distance from the Sun. To show this, Newton used the second and third Kepler's laws. But the end result, and the triumph of Newton, was to prove that the force of gravity, acting in accordance with its laws, leading to *elliptical orbits* for all the planets. In the following we will demonstrate this fact, putting aside the *Principia*, but following a simpler geometric-mechanical reasoning, due to the Nobel prize Richard Feynman.

Before continuing, it is helpful to recall the geometric definition of the ellipse as:
*An ellipse is a curve on a plane that surrounds two focal points such that the sum of the distances to the two focal points is constant for every point on the curve.*



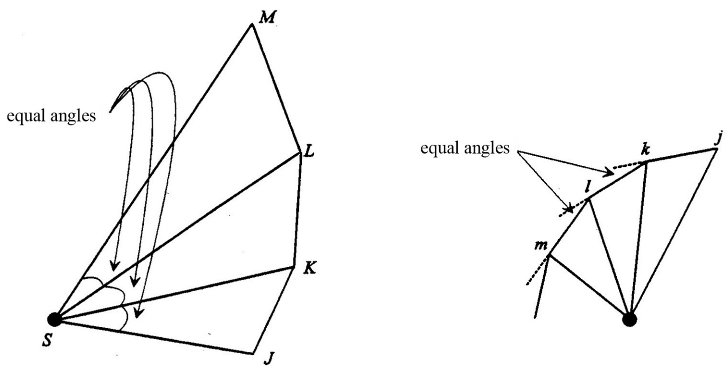
The distance from $F'$ to $F$ through $P$ is the same whatever the position of $P$.

In the Feynman's demonstration, is of central importance the following animated diagram (which I found on *Wikipedia*) that describes the geometric construction of the ellipse with the tangent method:




Feynman's contribution is essentially consisted, as we shall see, in the identification, in this figure, of the appropriate correspondences between geometric and mechanical elements of our problem.

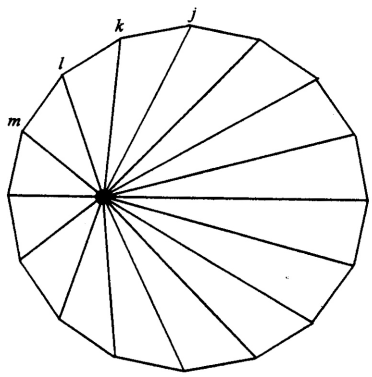
Feynman draws the orbit with a point sequence $J, K, L, M, N$, which no longer correspond to equal time intervals, as in the Newton diagram, but rather at equal angles respect to the departure position:



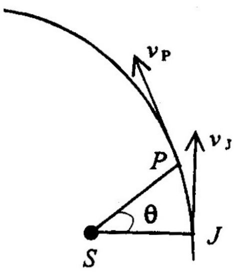

Equal angles mean that the triangles areas are not equal, but proportional to the square of the Sun distance [taken any two triangles, these are similar [1], for which their areas are to one another as the squares of respective dimensions]. So, since the areas are proportional to the times, the times necessary to describe these equal angles are proportional to the square of the distance. Now, being the velocity variations in equal times inversely

[1]This statement is evidently true for an infinitesimal subdivision of angles, that is, to the limit where the trajectory becomes a plane curve and the triangles are all isosceles with corresponding angles equal to each other.

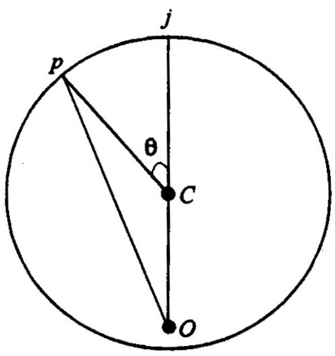
proportional to the square of the Sun distance, taking times proportional to the square of distance (equal angles) you get velocity variations all equal. Similarly to what was done above, we added in the above figure the velocity diagram, in which, for construction, are: $jk$ parallel to $KS$, $kl$ parallel $LS$, $lm$ parallel to $MS$, $jk = kl = lm$, and the outer angles are equal. The complete velocity diagram is a regular polygon, with the origin in an eccentric position:




By dividing the orbit in an ever more large number of segments, the velocity diagram gets closer and closer to a circle, with the velocity origin in an eccentric position. At this point Feynman plots the following two diagrams:




orbit diagram

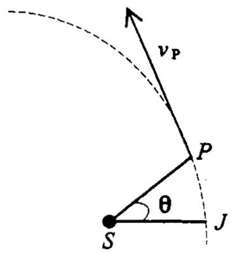


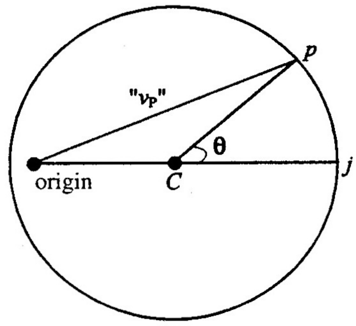

velocity diagram

In the orbit diagram, velocityes $vj$ and $vp$ are tangent to the curve in the $J$ and $P$ points. The velocity diagram will be a circle, with eccentric origin. The $J$ point in the orbit diagram is also the point closest to the Sun, where the orbital velocity has the highest value. The segment representing $vj$ in the velocity diagram, must then pass through the center of the circumference, because it must be the longest in the diagram. The $vp$ velocity is a segment from the origin parallel to $vp$. The $JSP$ and $jCp$ angles are equal, because the two diagrams are divided into the same number of equal angles. So, at every angle $\theta$ we know the direction of the tangent to the orbit that

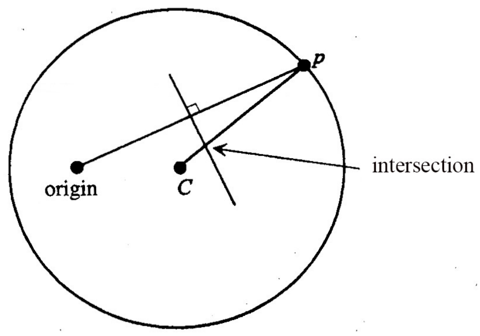
we construct.

Before proceeding to the the orbit construction, Feynman performs a 90° rotation of the velocity diagram, in the following way:

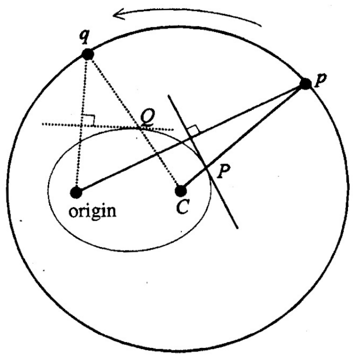



Now the two orbits, the left one in the figure and that to be built on the velocity diagram, are oriented in the same way. In particular, they are equally oriented: the direction of the segment joining the Sun to the $P$ point, and the direction of the tangent in $P$ to the orbit. To get the orbit, draw the axis of the $OP$ segment, that will be now parallel to the velocity $vp$, which is tangent to the orbit at the $P$ point:




But we know the direction of the belonging line of the orbital point $P$, that is the $Cp$ segment direction. Therefore, the intersection with $Cp$ of the traced axis, would enjoy of all the directional properties of the orbital point $P$ to be found, so:




while the point $p$ moves on the circumference up to $q$, the intersection point $P$ moves up to $Q$, and so on, giving origin to the orbit.
But this construction is exactly what we have shown in the previous animation, ie the geometric construction of the ellipse with the tangent method.

Therefore, we can claim to have realized the *shape* of the orbit we were looking for, showing that: ***the gravity force, acting in accordance with the Newton laws, generates elliptical orbits for all the planets***.

# 5 - The origin of the proof

The Feynman's demonstration is not new, it appears in the book *Matter and Motion*, published by James Clerk Maxwell in 1877. Maxwell attaches the proof to Sir William Rowan Hamilton, which was the first to make use of the velocity diagram (which he called hodograph) to study the motion of a body. Please read the following two pages of the above Maxwell's book and judge for yourself.

### 133. Kepler's Second Law

Law II.—The orbit of a planet with respect to the sun is an ellipse, the sun being in one of the foci.

Let $APQB$ (fig. 16) be the elliptic orbit. Let $S$ be the sun in one focus, and let $H$ be the other focus.

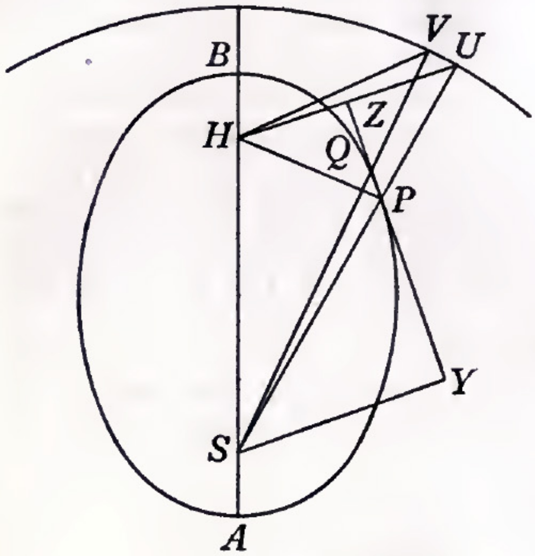



Fig. 16.

Produce $SP$ to $U$, so that $SU$ is equal to the transverse axis $AB$, and join $HU$, then $HU$ will be proportional and perpendicular to the velocity at $P$.

For bisect $HU$ in $Z$ and join $ZP$; $ZP$ will be a tangent to the ellipse at $P$; let $SY$ be a perpendicular from $S$ on this tangent.

If $v$ is the velocity at $P$, and $h$ twice the area swept out in unit of time, $h = vSY$.

Also if $b$ is half the conjugate axis of the ellipse

$$SY.HZ = b^2.$$

Now $$HU = 2HZ;$$

hence $$v = \frac{1}{2}\frac{h}{b^2}HU.$$

Hence $HU$ is always proportional to the velocity, and it is perpendicular to its direction. Now $SU$ is always equal to $AB$. Hence the circle whose centre is $S$ and radius $AB$ is the hodograph of the planet, $H$ being the origin of the hodograph.

The corresponding points of the orbit and the hodo-

graph are those which lie in the same straight line through $S$.

Thus $P$ corresponds to $U$ and $Q$ to $V$.

The velocity communicated to the body during its passage from $P$ to $Q$ is represented by the geometrical difference between the vectors $HU$ and $HV$, that is, by the line $UV$, and it is perpendicular to this arc of the circle, and is therefore, as we have already proved, directed towards $S$.

If $PQ$ is the arc described in [a very small] time, then $UV$ represents the acceleration [of velocity in that time;] and since $UV$ is on a circle whose centre is $S$, $UV$ will be a measure of the angular [movement in that time] of the planet about $S$. Hence the acceleration is proportional to the angular velocity, and this by Art. 129 is inversely as the square of the distance $SP$. Hence the acceleration of the planet is in the direction of the sun, and is inversely as the square of the distance from the sun.

This, therefore, is the law according to which the attraction of the sun on a planet varies as the planet moves in its orbit and alters its distance from the sun.